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ON THE SCATTERING EFFECT OF A ROUGH PLANE SURFACE

by

WILHELM MAGNUS

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ON THE SCATTERING EFFECT OF A ROUGH PLANE SURFACE

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CONTENTS

Abstract

A layer of dipoles is used to represent the roughness of a perfectly conducting plane surface. A random distribution of the dipoles is introduced by the assumption that the energy scattered by the dipoles shall show an additive behavior. The parameters of the problem are restricted by the assumption that the distance d between the source of radiation and the point of observation is large. On the other hand, the wavelength may be of the order of magnitude of the roughness, i.e., of the height σ of the scattering dipoles above the plane. Let h and η be the distances from the reflecting plane of the sending dipole and of the point of observation. Then the scattered energy is computed as a function of d, h, η, σ for horizontal and for vertical polarization. The formulas involve an undetermined factor which depends only on the density of the dipole distribution and on the ratio of the amplitudes of the incoming wave and of the wave emitted by a single scattering dipole

1. Notations

Let S be an electromagnetic point source and let R be a receiver, i.e. a point where the energy of the electric field emitted at S is measured. Let h and η be the distances of S and R from a plane perfectly conducting surface and let d be the horizontal distance between S and R. Let σ be the distance from the plane surface of a layer of scattering dipoles. The wavelength of electromagnetic waves shall be denoted by λ, and we define $K = 2\pi/\lambda$.

We assume that λ/h, λ/η , h/d, η/d are small numbers and that

$$(\sigma/\lambda)\,(h/d)^{1/2} \leqq 1, \qquad (\sigma/\lambda)(\eta/d)^{1/2} \leqq 1.$$

2. Introduction

The reflection of electromagnetic waves by a rough surface has been investigated from various points of view. Apart from papers in which the principles of geometrical optics are applied, (as for instance by Kerr [1]) the most important results seem to have been obtained by Ament [2], S. O. Rice [3] and V. Twersky [4]. Ament uses scattering dipoles in order to obtain a model for a rough surface. His approach, which at present is not fully accessible to the author, seems to be somewhat different from the approach employed in this report which also introduces scattering dipoles. Rice [3] considers comparatively smooth surfaces by assuming that the unevenness of the surface can be represented by a function f, such that both f and its derivatives are not too large. Twersky [4] derives exact solutions for scattered reflection from both a semicylindrical and hemispherical boss on an infinite plane and then obtains approximate solutions for surfaces consisting of either ordered or random configurations of bosses, small compared to the wavelength; the random cases being proposed as models for striated and rough surfaces.

If the rough surface is supposed to be e.g., the surface of the sea, and if the assumption is made that the main contribution to the scattering of the electromagnetic waves is due to the sea waves as a whole, the approach used by Twersky [4] seems to be the natural one. Also, the investigations carried through by Twersky have the advantage of being perfectly rigorous in the sense that they do not involve any unproved assumptions.

The present report introduces two basic assumptions. If the rough surface is supposed to be a model of the sea, the first assumption can be stated by saying that the scattering of the electromagnetic energy is mainly due to the peaks of the sea waves. This justifies the construction of a model in which the roughness of the surface is represented by a random distribution of scattering dipoles.

The second assumption is that "random distribution" is equivalent to "additive behavior of the scattered energy" in the following sense. The secondary or scattered waves produced by any two dipoles in the scattering layer are supposed to be incoherent. This means that no interference between them is possible and therefore that the energies transmitted to a point R by the scattering dipoles can simply be added at R. To prove this hypothesis it would be necessary to apply the methods developed by Reiche [5] in connection with the problem of the diffraction of light by vapor and gases. Reiche has solved this problem by a most remarkable and very successful method, which, however, if applied to the present case, would require an additional and very difficult investigation.

The present report is an attempt to study the scattering of electromagnetic energy under the assumptions stated above for sets of values of the geometrical and electrical parameters which seem to be likely to occur in practical problems. In particular, the restriction for σ/λ, (i.e. for the ratio of the height of the sea waves to the wavelength of the electromagnetic wave) is not a very severe one insofar as $\sigma/\lambda \leq 20$ can be admitted if the distance d between the point source S and the point of observation is large enough. It would be easy to supplement the results stated in this report by formulas which cover the case $d = 0$. In this case the point source and the place of observation would coincide and this would lead to integrals which can be evaluated by the same method as the one employed here. Actually, the integrals in (1.7), (2.2), and (3.2) simplify if we make $h = \eta$, $d = 0$.

With the notations explained in section 1 (cf. also Figure 1, 2 for the geometrical parameters), the results can be summarized as follows: The amount T of scattered electric energy at R is of the type

$$T = c_o Q\, I(k\sigma) d^{-v} \left\{ f(\tfrac{h}{d}) + f(\tfrac{\eta}{d}) \right\}.$$

In this formula, $c_0 Q$ is independent of σ, λ, η, d. Q may depend on k since Q characterizes both the density of the scattering dipoles and the interaction between the incoming wave and the dipoles. $I(k\sigma)$ is a constant for vertical polarization. For horizontal polarization, the behavior of $I(k\sigma)$ can easily be discussed, and it is about the same in the two-and three dimensional case. In two dimensions, $v = 1$, and in three dimensions, $v = 2$, $f(x) = x^2$ for horizontal polarization and $f(x) = -\log x$ for vertical polarization.

The two dimensional case has been studied as a model for a striated surface and also in order to make possible a comparison with the corresponding parts of the rigorous theory of Twersky.

The evaluation of the integrals (2.2) and (3.2) implies that for horizontal polarization the scattered energy arriving at R is mostly due to the parts of the surface which are close to S and to R. In the case of the vertical polarization, the whole stretch of the surface between S and R contributes about equally to the scattered energy.

It is of course likely that for practical purposes both horizontal and vertical scattering dipoles must be taken into account.

Altogether, it is the main purpose of this report to study the problem of the reflection from a rough surface under conditions which could prevail in practice. The simplifying assumptions on which the theory is based could be justified only by experiments. On the other hand, the formulas obtained are very simple and can be compared easily with experimental results.

3. The Two Dimensional Case

We consider a sender S and a receiver R which are represented by the points (o,h) and (d,η) in the x,y-plane. We assume that S is the source of a cylindrical wave with an axis perpendicular to the x,y-plane. The x-axis represents a perfectly conducting plane surface, and P denotes a source of a perturbing wave with the following properties. An electromagnetic wave with an amplitude E^* at P of its component perpendicular to the x,y-plane produces a secondary cylindrical wave with its center at P. The amplitude of the secondary wave shall be proportional to E^*, say qE^*

Let τ, σ be the coordinates of P (cf. Fig. 1)

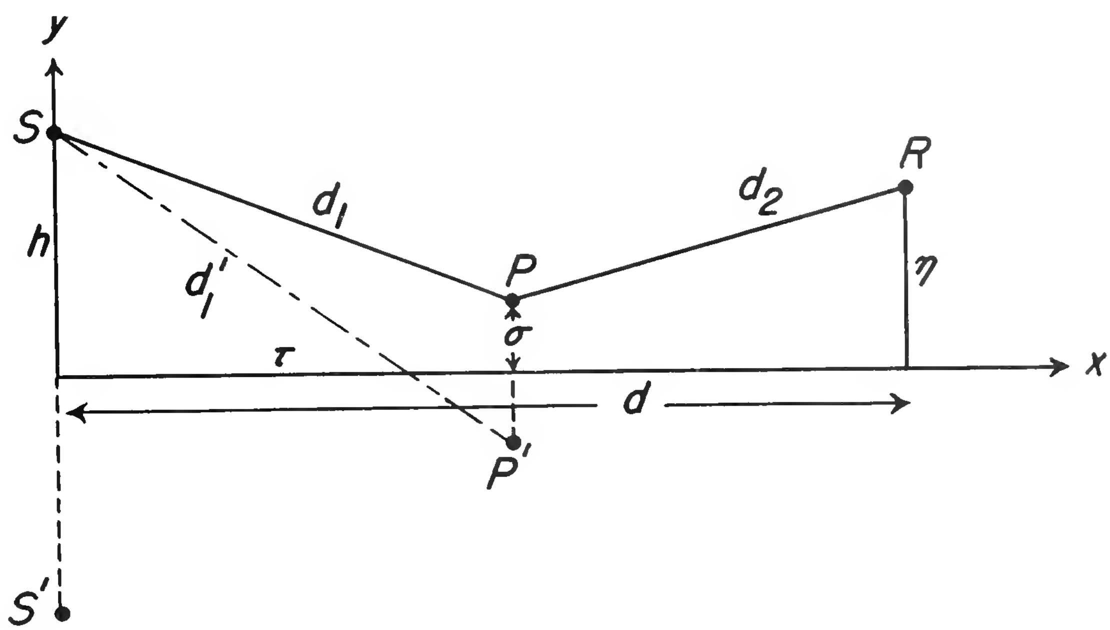

Figure 1

Let $K = \frac{2\pi}{\lambda}$, where λ is the wavelength; we shall assume that

(i) h, η are large compared with λ and σ.

(ii) d is large in comparison with h, η.

(iii) $\frac{\sigma}{\lambda} \left(\frac{h}{d}\right)^{1/2} \leqq 1, \quad \frac{\sigma}{\lambda} \left(\frac{\eta}{d}\right)^{1/2} \leqq 1.$

The following numerical quantities indicate ranges of values for h, η, d, σ which may be considered as typical examples:

$$\sigma \leqq 10\lambda$$
$$d \geqq 10{,}000\lambda$$
$$100\lambda \leqq h,\eta \leqq 1000\lambda$$

We wish to compute the electrical energy $T(\tau, \sigma)$ transmitted to R by the secondary wave produced at P. For this purpose, we take into account the presence of the perfectly conducting plane by adding a source at S' with the coordinates $(0, -h)$ with an amplitude which is the negative of that of the sender at S and by adding a second center of a perturbing wave at P' with the coordinates $(\tau, -\sigma)$.

Let $H(X)$ denote the function $H_0^{(2)}(kX)$, where $H_0^{(2)}$ is the second Hankel function of order zero. Let

$$(1.1) \qquad d_1 = \left[(h-\sigma)^2 + \tau^2\right]^{1/2}, \quad d_1' = \left[(h+\sigma)^2 + \tau^2\right]^{1/2}$$

$$(1.2) \qquad d_2 = \left[(\eta-\sigma)^2 + (d-\tau)^2\right]^{1/2}, \quad d_2' = \left[(\eta+\sigma)^2 + (d-\tau)^2\right]^{1/2}$$

denote the distances SP, SP', PR, P'R. The incoming field can be described as the field produced by the dipole at S and its specular image at S'. Since SP and S'P are large compared with λ, the resulting field at P will be

$$E_0\left(H(d_1) - H(d_1')\right)$$

This produces a secondary cylindrical wave with its center at P and with the amplitude

$$E_0\, q\, \left(H(d_1) - H(d_1')\right).$$

The presence of the reflecting surface will be taken into account by assuming that a source at P acts as if it was supplemented by a source with its center at P' (the specular image of P) with an amplitude of the same absolute value but the opposite sign. The electric field at R, which is produced by these two secondary waves, will then be

$$(1.3) \qquad E_0 q \left\{H(d_1) - H(d_1')\right\} \left\{H(d_2) - H(d_2')\right\},$$

where $\pm E_0$ is the amplitude of the primary waves with centers at S and S'.

Since kd_1, kd_1' etc., are much larger than 1, we may substitute the asymptotic forms, e.g.,

$$(1.4) \qquad (2/\pi)^{1/2}\,(ikd_1)^{-1/2}\,\exp(ikd_1)$$

for $H(d_1)$ in (1.3). Since h, η are large as compared to σ, we may write

$$(1.5) \qquad (h^2 + \tau^2)^{1/2}\left\{1 - \sigma h/(h^2 + \tau^2)\right\}$$

for d_1 and we may approximate $H(d_1)$ by

$$(1.6) \quad (2/\pi)^{1/2}\,(ik)^{-1/2}\,(h^2 + \tau^2)^{-1/4}\,\exp ik(h^2 + \tau^2)^{1/2}\,\exp\left(-ik\sigma h/(h^2 + \tau^2)^{1/2}\right).$$

Using the corresponding approximations for $H(d_1')$ etc., we find a comparatively simple expression for the square of the absolute value of the function in (1.6). This gives us $T(\tau, \sigma)$; the quantity we are interested in is the total amount $T_o(\sigma)$ of energy transmitted to R by a set of scattering dipoles distributed at random over a parallel to the x-axis. According to the postulate of the additive behavior of energy (which has been stated and explained in the introduction) we find

$$(1.7)\quad T_o(\sigma) = Q\,\frac{64}{(\pi k)^2}\int_{-\infty}^{+\infty} \frac{\left(\sin\dfrac{k\sigma h}{(\tau^2+h^2)^{1/2}}\,\sin\dfrac{k\sigma\eta}{((\tau-d)^2+\eta^2)^{1/2}}\right)^2}{(\tau^2+h^2)^{1/2}\left[(\tau-d)^2+\eta^2\right]^{1/2}}\,d\tau$$

where $Q = Q_o\, q^2\, E_o^2$ and where Q_o is a measure for the density of the scattering dipoles.

We shall not compute the exact value of $T_o(\sigma)$, giving merely an approximation for large values of d. We shall show that

$$(1.8)\qquad \lim_{d\to\infty} T_o\, d^3 = C_o$$

exists and is different from zero, and we shall determine C_o (as a function of $h, \eta, k\sigma$). For this purpose we observe that the integrand in (1.7) contributes to the integral essentially for those values of τ only for which either $|\tau|$ or $|d-\tau|$ is small compared with d, e.g. $|\tau| \leqq \sqrt{dh}$, $|\tau-d| \leqq \sqrt{d\eta}$. Now, for $|\tau| \leqq \sqrt{d}$ the integrand in (1.7) is practically (for large d)

$$(1.9)\qquad \left[(\tau^2+h^2)^{-1/2}\sin\frac{k\sigma h}{(\tau^2+h^2)^{1/2}}\right]\frac{(k\sigma\eta)^2}{d^3}$$

and therefore we may write approximately

$$(1.10)\quad T_o(\sigma) \approx \frac{(k\sigma\eta)^2}{d^3}\cdot\frac{64Q}{(\pi k)^2}\int_{-\infty}^{+\infty}\sin^2\left\{k\sigma h(\tau^2+h^2)^{-1/2}\right\}(\tau^2+h^2)^{-1/2}\,d\tau$$

$$+\frac{(k\sigma h)^2}{d^3}\cdot\frac{64Q}{(\pi k)^2}\int_{-\infty}^{+\infty}\sin^2\left\{k\sigma\eta\left((\tau-d)^2+\eta^2\right)\right\}\frac{d\tau}{((\tau-d)^2+\eta^2)^{1/2}}$$

If we introduce new variables in the first and the second integral in (1.10), $\rho = h\tau$ and $\rho = \eta(\tau - d)$, respectively, we find

$$(1.11) \quad T_0(\sigma) \approx \frac{64Q}{(\pi k)^2} \; \frac{(k\sigma)^2 \, (h^2+\eta^2)}{d^3} \quad I_0 \; ,$$

where

$$(1.12) \qquad I_0 = \int_{-\infty}^{+\infty} \sin^2 \frac{k\sigma}{(1+\rho^2)^{1/2}} \; \frac{d\rho}{(1+\rho^2)^{1/2}} \; .$$

Substituting

$$(1.13) \qquad \phi = (1+\rho^2)^{-1/2} \; , \qquad \frac{d\phi}{d\rho} = -\rho\,(1+\rho^2)^{-3/2} \; ,$$

we obtain from (1.12)

$$(1.14) \qquad I_0 = 2\int_0^1 \frac{\sin^2 k\sigma\phi}{\phi(1-\phi^2)^{1/2}}\, d\phi = \int_0^1 \frac{1-\cos 2k\sigma\phi}{\phi\,(1-\phi^2)^{1/2}}\, d\phi$$

$$= \frac{\pi}{2}\int_0^{2k\sigma} \mathbf{H}_0(x)\, dx$$

Here $\mathbf{H}_0$ denotes Struve's function of order zero (cf. Watson [6]); $\mathbf{H}_0(x)$ has the asymptotic expansion

$$(1.15) \quad \mathbf{H}_0(x) = Y_0(x) + \frac{2}{\pi x}\left(1 - \frac{\frac{1}{2}\cdot 2!}{1!\; x^2} + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot 4!}{2!\; x^4} \mp \ldots\right)$$

where $Y_0(x)$ is the Bessel function of the second kind (Neumann's function) of order zero. Since

$$(1.16) \qquad \int_0^{\infty} Y_0(x)\, dx = 0 \; ,$$

we find from (1.14) and (1.15) that for large values of $2k\sigma$

$$(1.17) \qquad I_0 \approx \log 2k\sigma \; ,$$

in the sense that

$$(1.18) \qquad \lim_{\sigma\to\infty} \frac{I_0}{\log 2k\sigma} = 1 \; .$$

We find from (1.14) by expanding the integrand in a series of powers of ϕ and by integrating term by term

$$(1.19)\quad I_o = (2k\sigma)^2 \sum_{m=o}^{\infty} \frac{(-16k^2\sigma^2)^m \; m!\; m!}{(2m+2)!\;(2m+1)!} = 2k^2\sigma^2 - \frac{4}{9}k^4\sigma^4 + \frac{128}{2025}k^6\sigma^6 \pm \dots$$

This series can be used for a computation of I_o for values of $2k\sigma$ which are not too large.

According to the remarks made before (1.3), the energy transmitted directly from S to R is approximately (for $|E_o| = 1$)

$$(1.20)\quad \frac{8}{\pi kd}\sin^2(k\eta h/d) \approx |H(\overline{SR}) - H(\overline{S'R})|^2$$

If d is so large that we may substitute $k\eta h/d$ for $\sin(k\eta h/d)$, we find for the ratio of scattered energy to directly transmitted energy

$$(1.21)\quad \frac{8Q}{\pi k}(k\sigma)^2 \; I_o \left\{(kh)^{-2} + (k\eta)^{-2}\right\}$$

where I_o is a function of $k\sigma$ and is given by (1.14), (1.19). Q depends on the density of the distribution of scattering dipoles, on their interaction with an incoming field and therefore (possibly) on k, but not on σ, h or η.

4. The Three Dimensional Case, Horizontal Polarization

We introduce Cartesian coordinates x,y,z in space. Let the x,y plane be the reflecting surface, and let (o,o,h) and (o,d,η) be the coordinates of a sending dipole S and a receiving dipole R, both of which are perpendicular to the (y,z) plane. Let (β, τ, σ) be the coordinates of a "disturbing" dipole at a height σ over the x,y-plane. The axis of this dipole shall also be parallel to the x_+-axis.

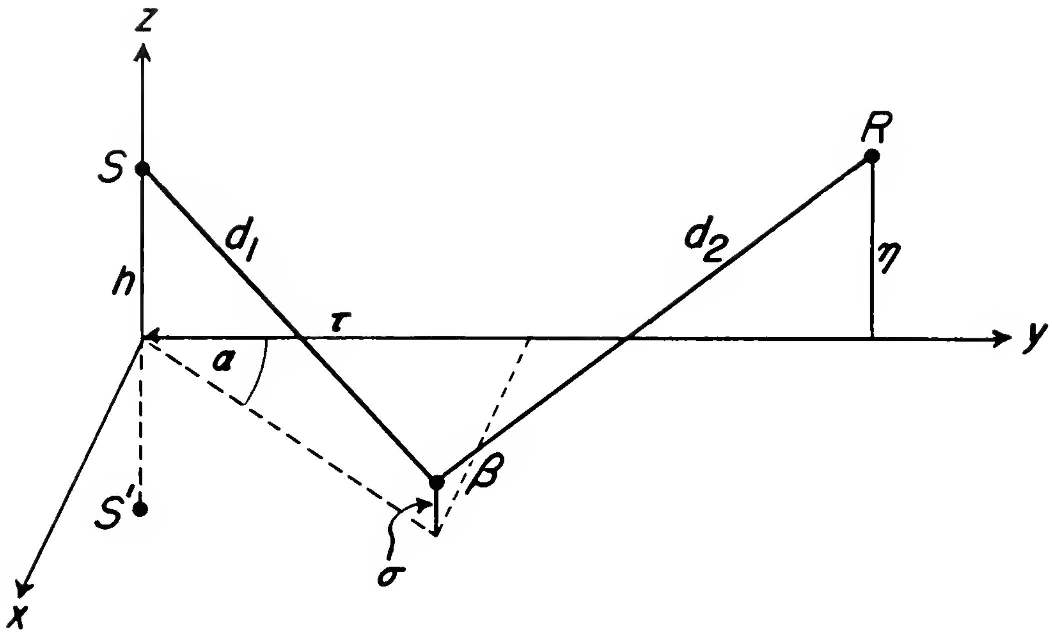

Figure 2

We may proceed with the computation of the scattered electric energy in the same manner as in the two-dimensional case, apart from the two following considerations. Firstly, we must introduce for $H(x)$ the function

$$(kx)^{-1} \exp ikx$$

and secondly, we must observe that the electric field produced by S at the place of the disturbing dipole is now (cf. Fig. 2 for the notations)

$$(2.1) \qquad H(d_1) \cos\alpha .$$

We then obtain for the total amount $T_1(\sigma)$ of scattered electric energy at R the formula

$$(2.2) \quad T_1(\sigma) = \frac{16Q}{k^4} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\left[\sin(k\sigma h/W_1)\sin(k\sigma\eta/W_2)\right]^2 \tau^2(\tau-d)^2 \, d\tau d\beta}{W_1^2 W_2^2 (\tau^2+\beta^2) \left[(d-\tau)^2+\beta^2\right]}$$

where

$$(2.3) \qquad W_1 = (\tau^2+\beta^2+h^2)^{1/2} , \qquad W_2 = \left[(d-\tau)^2+\beta^2\eta^2\right]^{1/2} .$$

Making the same assumptions about the relative order of magnitude of $k\sigma$, h, η, d, as in the preceding section, we find again that the integral in (2.2) may be approximated by the sum of two integrals which arise from (2.2) by substituting d

for W_1 or for W_2. To substitute d for W_1 implies that one essential contribution to $T_1(\sigma)$ is due to an integration over a certain domain in the τ, β -plane which does not extend very far beyond a neighborhood of $\tau = \beta = 0$. In this neighborhood, W_2 is practically equal to d and $(\tau - d)^2 \Big/ \left[(\tau - d)^2 + \beta^2\right]$ is practically equal to 1. The other "essential" contribution to T_1 will be obtained by integrating over a neighborhood of $\tau = d$, $\beta = 0$, and this implies $W_1 \approx d$, $\tau^2/(\tau^2 + \beta^2) \approx 1$.

Substituting d for W_2 in (2.2) gives the integral

$$(2.4) \quad \frac{16Q}{k^4} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left(\frac{k\sigma\eta}{d}\right)^2 d^{-2} \; \frac{\sin^2(k\sigma h/W_1)}{W_1^2} \; \frac{\tau^2}{\tau^2 + \beta^2} \, d\tau \, d\beta \, .$$

If we introduce polar coordinates r, Θ by

$$(2.5) \qquad \tau = hr \cos \Theta \, , \qquad \beta = hr \sin \Theta \, ,$$

(2.4) becomes

$$(2.6) \quad \frac{16Q}{k^4} \int_0^{2\pi} \cos^2 \Theta \, d\Theta \int_0^{\infty} \left(\frac{k\sigma\eta}{d^2}\right)^2 \frac{\sin^2\left(k\sigma(r^2+1)^{-\frac{1}{2}}\right) r}{r^2+1} \, dr$$

$$= \frac{16\pi Q}{(kd)^4} \, \eta^2 \, (k\sigma)^2 \int_0^{\infty} r \sin^2\left(k\sigma(r^2+1)^{-\frac{1}{2}}\right) \; (r^2+1)^{-1} \, dr \, .$$

Let $\rho = (r^2 + 1)^{-1/2}$, and let $C = 0.577215 \ldots$ be the Euler-Mascheroni constant. Then the integral in (2.6) becomes

$$(2.8) \quad \int_0^1 \rho^{-1} \sin^2 k\sigma\rho \, d\rho = \frac{1}{2} \int_0^1 (1 - \cos 2k\sigma\rho)\rho^{-1} \, d\rho$$

$$= \frac{1}{2} \int_0^{2k\sigma} (1 - \cos t) \, t^{-1} \, dt = \frac{1}{2} C + \frac{1}{2} \log 2k\sigma - \frac{1}{2} \operatorname{Ci}(2k\sigma),$$

where

$$(2.9) \qquad \operatorname{Ci}(x) = \int_x^{\infty} \frac{\cos t}{t} \, dt$$

denotes the integral cosine and where

$$(2.10) \qquad \frac{1}{2} C = 0.288607 \cdots .$$

Since $Ci(x) \to 0$ as $x \to \infty$, (2.8) gives the asymptotic behavior of the integral (2.6) for large $k\sigma$.

Applying the same formulas to the integral obtained from (2.4) by substituting d for W_1, we find the approximate expression

$$T_1(\sigma) \approx \frac{16\pi Q}{(kd)^4} (k\sigma)^2 \ I_1(k\sigma) \ (\eta^2 + h^2) \,. \tag{2.11}$$

where

$$I_1(k\sigma) = 0.288607 + \frac{1}{2} \log 2k\sigma - \frac{1}{2} Ci \ (2k\sigma) \,. \tag{2.12}$$

$$= 2k^2\sigma^2 \sum_{m=0}^{\infty} \frac{(-4k^2\sigma^2)^m}{(2m+2)!\,(2m+2)}$$

3. The Three Dimensional Case,

Vertical Polarization.

We assume now that both the sending dipole at S and the scattering dipoles in Fig. 2 are vertical. In this case we may take $\sigma = 0$, since the undisturbed reflecting plane does not extinguish the electric field at its surface. Let α' be the angle which the line labeled d_1 encloses with the x,y-plane. Then

$$(3.1) \qquad \cos^2\alpha' = \frac{\tau^2 + \beta^2}{\tau^2 + \beta^2 + h^2}$$

and we find for the scattered energy at R

$$(3.2) \qquad T_2 = \frac{4Q}{k^4} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{(\tau^2 + \beta^2)\left[(d-\tau)^2 + \beta^2\right] d\tau\, d\beta}{(\tau^2 + \beta^2 + h^2)^2\left[(d-\tau)^2 + \beta^2 + \eta^2\right]^2}$$

We shall integrate with respect to β first. For this purpose, we introduce

(3.3) $|\tau| = p$, $|d-\tau| = q$, $(\tau + h^2)^{\frac{1}{2}} = u$, $\left((d-\tau)^2 + \eta^2\right)^{\frac{1}{2}} = v$, and we compute

$$(3.4) \qquad J = \int_{-\infty}^{+\infty} \frac{(p^2 + \beta^2)(q^2 + \beta^2)}{(u^2 + \beta^2)^2\,(v^2 + \beta^2)^2}\, d\beta \ .$$

J can be evaluated by computing the sum of the residues of the integrand in (3.4) in the upper half of the complex β-plane.

After an elementary calculation we find for the sum of these residues, multiplied by $2\pi i$

$$(3.5) \qquad J = \frac{\frac{1}{2}\pi}{(u+v)^3}\left\{ \frac{(uv + p^2)\,(uv + q^2)}{u^2\, v^2} + \frac{(pq)^2\,(u+v)^2}{(uv)^3} \right\}$$

Since u,v,p,q depend on τ, $J = J(\tau)$ is a function of τ, and we have now to compute

$$(3.6) \qquad \int_{-\infty}^{+\infty} J(\tau)\, d\tau \ .$$

We shall simplify this problem by substituting u for p and v for q. This will affect the value of the integral in (3.2), but it can be shown that the relative errors due to these substitutions are of the order of magnitude of h/d and η/d. Therefore we find

$$(3.7)\qquad \int_{-\infty}^{+\infty} J(\tau)\, d\tau \approx \pi \int_{-\infty}^{+\infty} \frac{d\tau}{(u+v)\,uv}$$

Multiplying by $u - v$ in the numerator and in the denominator of the integrand in the right hand side of (3.7), we find from (3.3),

$$(3.8)\qquad \int_{-\infty}^{+\infty} J(\tau)\, d\tau \approx \int_{-\infty}^{+\infty} \left(\frac{1}{\left[(d-\tau)^2 + \eta^2\right]^{1/2}} - \frac{1}{(\tau^2+h^2)^{1/2}} \right) \frac{d\tau}{2d\tau - d^2 - \eta^2 + h^2}$$

This is an elementary integral. The singularity at the second factor of the integrand at

$$(3.9)\qquad \tau = A = \frac{d^2 + \eta^2 - h^2}{2d}$$

is compensated for by the zero of

$$(3.10)\qquad \left[(d-\tau)^2 + \eta^2\right]^{-1/2} - (\tau^2 + h^2)^{-1/2}$$

at $\tau = A$. Now we have that the indefinite integral

$$(3.11)\qquad \int \frac{d\tau}{(\tau - A)\,(\tau^2+h^2)^{1/2}} = \frac{1}{B} \log \frac{\phi - \theta_1}{\phi - \theta_2} ,$$

where

$$(3.12)\qquad B^2 = A^2 + h^2 = (2d)^{-2} \left\{ (d^2 + \eta^2 + h^2)^2 - 4\eta^2 h^2 \right\}$$

$$(3.13)\qquad \theta_1 = \frac{1}{h}(A + B), \quad \theta_2 = \frac{1}{h}(A - B)$$

$$(3.14)\qquad \phi = \frac{1}{h}\left(\tau + \sqrt{\tau^2 + h^2} \right) .$$

Since

$$(3.15)\qquad \frac{d\phi}{d\tau} = h^{-1}\left(1 + \tau(\tau^2 + h^2)^{-1/2}\right)$$

is positive for $-\infty < \tau < \infty$, we see that ϕ runs exactly once from 0 to ∞ if τ runs from $-\infty$ to $+\infty$. In this interval the right hand side of (3.11) becomes singular only at $\phi = \theta_1$.

Similarly, we have

$$(3.16)\quad \int \frac{d\tau}{(\tau - A)\left[(\tau - d)^2 + \eta^2\right]^{1/2}} = \int \frac{d\tau^*}{(\tau^* - A^*)\,(\tau^{*2} + \eta^2)^{1/2}}$$

$$= \frac{1}{B} \log \frac{\phi^* - \theta_1^*}{\phi^* - \theta_2^*}$$

where

$$(3.17)\quad A^* = A - d, \qquad \theta_1^* = \frac{1}{\eta}(A^* + B), \quad \theta_2^* = \frac{1}{\eta}(A^* - B),$$

$$(3.18)\quad \phi^* = \frac{1}{\eta}\left(\tau^* + \sqrt{\tau^{*2} + \eta^2}\right).$$

Combining (3.11) and (3.16), we find from (3.8)

$$(3.19)\quad \int_{-\infty}^{+\infty} J(\tau)\,d\tau = \frac{+1}{2dB} \log \frac{\theta_1}{\theta_1^*}\,\frac{\theta_2^*}{\theta_2} = \frac{+1}{2dB} \log \frac{(A+B)(A^*-B)}{(A-B)(A^*+B)}.$$

Since

$$(3.20)\quad A^2 + h^2 = A^{*2} + \eta^2 = B^2$$

and since $A, -A^*, B$ all have the order of magnitude of $\frac{1}{2}d$, we can approximate the right hand side of (3.19) by

$$(3.21)\quad \frac{1}{d^2} \log \frac{d^4}{\eta^2 h^2} = 2d^{-2} \log \frac{d^2}{h\eta};$$

the relative error will be of the order of magnitude of $\frac{h}{d} + \frac{\eta}{d}$. This gives the approximate result for the scattered energy

$$(3.22)\quad T_2 \approx \frac{8Q}{k^4\, d^2} \log \frac{d^2}{h\eta}.$$

References

1. Kerr, Donald E., Propagation of Short Radio Waves. (M.I.T. Radiation Laboratory series). Section 5.4., p. 411-418.

2. Rice, S. O., Wave Propagation over Rough Surfaces. Communications on Pure and Applied Mathematics, Vol.4., 1951.pp. 351-378.

3. Ament, W. S.,Naval Research Laboratory Branch Memorandum WPR TA 3/50 March 8, 1950 (Unpublished)

4. Twersky, V., On the Non-Specular Reflections of Electromagnetic Waves. Journal of Applied Physics, 22, p. 825, 1951. An extensive list of references is given there. Also: Mathematics Research Group, Research Report No. EM-26, January 1951.

5. Reiche, Fritz, Zur Theorie der Dispersion in Gasen and Daempfen. Annalen der Physik, 4. Folge, <u>50</u> , 1-52, 121-162, (1916).

6. Watson, G. N., A Treatise on the Theory of Bessel Functions, Cambridge, 1922.

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